SPEEDUPS OF ERGODIC GROUP EXTENSIONS

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We dedicate this paper to the memory of our great friend and teacher, Dan Rudolph.

Abstract. We prove that for all ergodic extensions S_1 of a transformation by a locally compact second countable group G, and for all $G-$extensions S_2 of an aperiodic transformation, there is a relative speedup of S_1 that is relatively isomorphic to S_2. We apply this result to give necessary and sufficient conditions for two ergodic $n-$point or countable extensions to be related in this way.

1. Introduction

Let $(X, \mathcal{B}, \mu)$ be a Lebesgue probability space and $T : X \to X$ an ergodic $\mu-$preserving automorphism. By a *speedup* of T we mean an automorphism of X of the form $x \mapsto T^{p(x)}(x)$, where p is a positive integer-valued function on X. We denote such an automorphism by T^p. It is natural to ask which automorphisms (up to isomorphism) can be obtained from T in this way. If p is integrable there are significant restrictions on the possible speedups of T. It was proved in [N], for example, that if S is isomorphic to $T^{p(x)}$ and $\int p d\mu < \infty$, then the entropies of S and T satisfy $h(S) = \left(\int p d\mu\right) h(T)$. As another example, from [OW] we see that if S is isomorphic to $T^{p(x)}$ and $\int p d\mu < \infty$, then T is a factor of a finite measure preserving transformation that induces S. Thus, if S is loosely Bernoulli, it can only be expressed as an integrable speedup of T if T is also loosely Bernoulli. However, if p is not required to be integrable, then there are no obstructions to this relation; in [AOW] the general result was proved that for all ergodic finite measure preserving automorphisms T and all aperiodic finite measure preserving S, there is a speedup of T that is isomorphic to S. In this paper we prove a conditional version of that result in the case of ergodic group extensions, and we give an application of this result to the classification of ergodic finite extensions.

Suppose that $(X, \mathcal{B}, \mu)$ and T are as above, and T is a factor of an automorphism S of the space $(Y, \mathcal{C}, \nu)$. Then by a speedup of S *relative to* T we mean a speedup S^p where p is measurable with respect to the factor $(X, \mathcal{B}, \mu)$. Of particular interest to us it the case where S is a group extension of T by a locally compact second countable group G. We recall some basic definitions:

Let G be a locally compact second countable group with left Haar measure λ, and let $\sigma : X \times \mathbb{Z} \to G$ be a cocycle for T. That is, σ is a measurable function such

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that for almost all $x \in X$ and all $n, m \in \mathbb{Z}$

$$\sigma(x, m+n) = \sigma(T^n x, m)\,\sigma(x, n)\,. \tag{1.1}$$

From T and σ we obtain an automorphism T_σ of $(X \times G, \mu \times \lambda)$ such that for all $n \in \mathbb{Z}$,

$$(T_\sigma)^n(x, g) = (T^n x, \sigma(x, n)\, g)$$

which has (T, X) as a factor. We refer to the map T_σ as a $G-$extension of T, and to (T, X) as the base factor of the extension. We write $\sigma^{(n)}$ for the function

$$\sigma^{(n)} : x \mapsto \sigma(x, n)$$

and note that σ is determined by the function $\sigma^{(1)}$, because of condition (1.1).

Given a cocycle σ for T and a measurable function $\alpha : X \to G$ we obtain a new cocycle for T, which we denote by σ^α, by setting

$$\sigma^\alpha(x, n) = \alpha(T^n x)\,\sigma(x, n)\,(\alpha(x))^{-1}\,.$$

Thus, given two functions $\alpha, \beta : X \to G$, we have

$$(\sigma^\alpha)^\beta = \sigma^{\beta\alpha}.$$

Two cocycles σ and σ' for T are said to be cohomologous if there exists a measurable function $\alpha : X \to G$ such that $\sigma' = \sigma^\alpha$. In this case we say σ' is cohomologous to σ by the transfer function α.

We say that two $G-$extensions T_σ and $T'_{\sigma'}$ on spaces X and X' are $G-$isomorphic if there is an isomorphism $\Phi : X \times G \to X' \times G$ between T_σ and $T'_{\sigma'}$ of the form

$$\Phi(x, g) = (\phi(x), \alpha(x)\, g)$$

where $\phi : X \to X'$ is an isomorphism between T and T' and $\alpha : X \to G$ is a measurable function. This is the case precisely when there is an isomorphism ϕ between T and T' such that the cocycle $\sigma'\phi$ for T is cohomologous to σ by the transfer function α, where $\sigma'\phi$ is given by

$$\sigma'\phi(x, n) = \sigma'(\phi x, n)\,.$$

Given a $G-$extension T_σ we consider speedups of T_σ relative to the base factor (T, X). Each such relative speedup of T_σ determines and is determined by a speedup of the factor T. Thus if $(T_\sigma)^p$ is a relative speedup of T_σ, we have $(T_\sigma)^p = (T^p)_{\sigma^{(p)}}$, where $\sigma^{(p)}$ is the cocyle for T^p determined by the values

$$\sigma^{(p)}(x, 1) = \sigma(x, p(x))\,.$$

Our first goal is to prove the following theorem:

Theorem 1. *Let T_σ and $T'_{\sigma'}$ be $G-$extensions of $(T, X, \mathcal{B}, \mu)$ and $(T', X', \mathcal{B}', \mu')$, where G is a locally compact second countable group. Suppose that T_σ is ergodic and T' is aperiodic. Let U be a neighborhood of e_G. Then there is a relative speedup of T_σ which is $G-$isomorphic to $T'_{\sigma'}$ by a $G-$isomorphism whose transfer function α satisfies $\alpha(x) \in U$ a.e.*

We remark that, as shown by Herman and Zimmer ([H], [Z]), a locally compact second countable group G admits ergodic $G-$extensions if and only if it is amenable.

Our original proof of this theorem in the case of ergodic $G-$extensions for compact G [BF] used techniques derived from the restricted orbit equivalence theory of Rudolph and Kammeyer [R],[KR1],[KR2], (and so ultimately from Ornstein's

isomorphism theorem [O]). However, as the referee has generously pointed out, a far simpler proof is available using the methods to be found in [AOW], which yields a stronger result, and we present that argument here.

We note that theorem 1 may be viewed as an analogue of the orbit equivalence result for $G-$extensions obtained in [F], which was also obtained by other methods in [G]. The point of view which we take here has much in common with that of Golodets and Sinel'shchikov in their paper [GS], which deals with questions of orbit equivalence. In [G] a classification of finite extensions up to factor orbit equivalence was given, and after proving theorem 1 we will adapt the methods of [G] to give an analogous classification of finite extensions with respect to the speedup relation we have introduced here.

2. Technical preliminaries

We use the following terminology. A *Rokhlin tower* $\mathcal{T}$ (or simply *tower*) for an automorphism T on $(X, \mathcal{B}, \mu)$ is a pairwise disjoint collection $\{A_i\}_{i=1}^{h}$ of measurable sets in X such that for each i, $T(A_i) = A_{i+1}$. Each $A_i \in \mathcal{T}$ is called a *level* of $\mathcal{T}$, A_1 is the *base*, $h = h(\mathcal{T})$ is the *height*, and the common value $\mu(A_i)$ is the *width* $w_{\mathcal{T}}$ of $\mathcal{T}$. We let $|\mathcal{T}| = \bigcup_{i=1}^{h} T^i A_1$ and $|\mathcal{T}|^o = \bigcup_{i=1}^{h-1} T^i A_1$. A *column* of $\mathcal{T}$ is a tower of the form $\left\{T^i B\right\}_{i=0}^{h-1}$, where B is a measurable subset of the base of $\mathcal{T}$.

A *castle* for T is a finite collection $\mathcal{C} = \{\mathcal{T}_j\}_{j=1}^{J}$ of towers for T such that $|\mathcal{T}_{j_1}| \cap |\mathcal{T}_{j_2}| = \emptyset$ for all $j_1 \neq j_2$. We let $|\mathcal{C}| = \bigcup_{j=1}^{J} |\mathcal{T}_j|$ and $|\mathcal{C}|^o = \bigcup_{j=1}^{J} |\mathcal{T}_j|^o$. We refer to $X \setminus |\mathcal{C}|$ as the *residual* set of $\mathcal{C}$. A *level* of $\mathcal{C}$ (respectively a *column* of $\mathcal{C}$) is a level (resp. column) of a tower in $\mathcal{C}$. Thus $\bigcup \{\mathcal{T} : \mathcal{T} \in \mathcal{C}\}$ is the set of all levels of $\mathcal{C}$, which we denote by $L(\mathcal{C})$.

If $\mathcal{T}$ is a tower for T then each finite measurable partition $\mathcal{Q} = \{B_j\}_{j=1}^{J}$ of the base of $\mathcal{T}$ gives rise to a castle $\mathcal{T}_Q$ whose towers are the columns of $\mathcal{T}$ with bases B_j. Given a finite partition $\mathcal{P}$ of $|\mathcal{T}|$, we obtain a partition $\mathcal{P}_{\mathcal{T}}$ of the base B of $\mathcal{T}$ whose atoms are maximal sets B_j such that for every $i \in \{1, 2, ..., h_{\mathcal{T}}\}$, $T^i B_j$ is contained in a single atom of $\mathcal{P}$. That is, $\mathcal{P}_{\mathcal{T}}$ is the trace of $\bigvee_{i=0}^{h-1} T^{-i}\mathcal{P}$ on B. This partition yields a castle $(\mathcal{T})_{\mathcal{P}_{\mathcal{T}}}$ as above. We refer to this castle as the castle of $\mathcal{P}-$columns of $\mathcal{T}$. We make similar definitions for castles $\mathcal{C}$ and partitions of $|\mathcal{C}|$ or of the bases of the towers of $\mathcal{C}$. We let $\mathcal{P}(\mathcal{C})$ denote the partition of X into the levels of $\mathcal{C}$ and the residual set of $\mathcal{C}$.

Given two castles $\mathcal{C}_1$ and $\mathcal{C}_2$ for T, we write $\mathcal{C}_1 \leq \mathcal{C}_2$ if $\mathcal{C}_2$ can be viewed abstractly as having been obtained from $\mathcal{C}_1$ by a cutting and stacking construction, as in [AOW]. More formally, this means the following: (i) $|\mathcal{C}_1| \subset |\mathcal{C}_2|^o$, (ii) There is a finite partition Q of the bases of the towers of $\mathcal{C}_1$ such that each level of the castle $(\mathcal{C}_1)_Q$ is a level of $\mathcal{C}_2$, and (iii) for each tower of $(\mathcal{C}_1)_Q$ there is a tower of $\mathcal{C}_2$ that contains it. Note that condition (i) implies that if $\{A_i\}_{i=1}^{h_2}$ is a tower in $\mathcal{C}_2$ and A_j is a base of a tower of $(\mathcal{C}_1)_Q$ of height h_1, then we must have $j \leq h_2 - h_1$.

We make use of the following lemmas. Lemmas 1 and 2 are well known, but we include their proofs for the convenience of the reader.

Lemma 1. *If T_σ is an ergodic $G-$extension of (T, X, μ), then given sets $A, B \subset X$ of positive measure, and a non-empty open set $U \subset G$, there are a set $A' \subset A$ of positive measure and $n' \in \mathbb{N}$ such that $T^{n'}(A') \subset B$ and for all $x \in A'$, $\sigma(x, n') \in U$.*

Proof. Fix sets $A, B \subset X$ of positive measure, and a non-empty open set $U \subset G$. Choose non-empty open sets V_0 and V_1 in G so that $e_G \in V_0$ and $V_1 V_0^{-1} \subset U$. Since T_σ is ergodic, for almost every $(x, g) \in A \times V_0$, there are (infinitely many) $n \in \mathbb{N}$ such that

$$(T_\sigma)^n (x, g) \in B \times V_1.$$

Hence there exists some $g_0 \in V_0$ such that for almost all $x \in A$, there exists $n \in \mathbb{N}$ such that

$$(T_\sigma)^n (x, g_0) \in B \times V_1.$$

For each $n \in \mathbb{N}$, let $A_n = \{x \in A : (T_\sigma)^n (x, g_0) \in B \times V_1\}$. Then for some $n' \in \mathbb{N}$, $\mu(A_{n'}) > 0$, and so for each $x \in A_{n'}$ we have

$$\sigma(x, n') g_0 \in V_1$$

so

$$\sigma(x, n') \in V_1 g_0^{-1} \subset U.$$

Setting $A' = A_{n'}$ we have the desired result. $\square$

For A' B and n' satisfying the conclusions of this lemma, we say (A', n') is $(B, U)-$good, or simply A' is $(B, U)-$good. We strengthen this lemma to obtain the following lemma.

Lemma 2. *If T_σ is an ergodic $G-$extension of (T, X, μ), then, given $A, B \subset X$ of equal measure and a non-empty open set $U \subset G$, there is a measurable function $p : A \to \mathbb{N}$ such that $T^p \restriction A$ is an isomorphism from A to B and, for almost every $x \in A$, $\sigma(x, p(x)) \in U$.*

Proof. Fix $A, B \subset X$ of equal measure and non-empty open $U \subset G$. Fix $\varepsilon_i \downarrow 0$. Let

$$a_1 = \sup\{\mu(A') : A' \subset A \text{ and } A' \text{ is } (B, U) - \text{good}\}.$$

Choose $A_1 \subset A$ and $n_1 \in \mathbb{N}$ such that (A_1, n_1) is $(B, U)-$good and $\mu(A_1) > a_1 - \varepsilon_1$. If $\mu(A_1) = \mu(A)$, we are done. If $\mu(A_1) < \mu(A)$, let

$$a_2 = \sup\{\mu(A') : A' \subset A \backslash A_1 \text{ and } A' \text{ is } (B \backslash (T^{n_1} A_1), U) - \text{good}\}$$

and choose $A_2 \subset A \backslash A_1$ and $n_2 \in \mathbb{N}$ such that (A_2, n_2) is $(B \backslash T^{n_1} A_1, U)-$good and $\mu(A_2) > a_2 - \varepsilon_2$.

Continue in this way to obtain a pairwise disjoint sequence $\{A_i\}$ and integers $n_i \in \mathbb{N}$. If, for some $k \in \mathbb{N}$, $\mu\left(\bigcup_{i=1}^k A_i\right) = \mu(A)$, we are done. Suppose, then that for all k, $\mu\left(\bigcup_{i=1}^k A_i\right) < \mu(A)$. If in fact $\mu\left(\bigcup_{i=1}^\infty A_i\right) < \mu(A)$, then by Lemma 1 there is a set $A' \subset A \backslash \left(\bigcup_{i=1}^\infty A_i\right)$ of positive measure and $n' \in \mathbb{N}$ such that (A', n') is $(B, U)-$good. But $\sum_{i=1}^\infty \mu(A_i) < \infty$, so $\mu(A_i) \to 0$, so $\mu(A_i) + \varepsilon_i \to 0$, so for some i

$$a_i < \mu(A_i) + \varepsilon_i < \mu(A')$$

which contradicts the choice of a_i. Hence $\mu\left(\bigcup_{i=1}^\infty A_i\right) = \mu(A)$, and we are done in this case as well. $\square$

We note that this lemma can easily be strengthened to say that if a measurable function $p_1 : A \to \mathbb{N}$ is given, then the function p can be chosen so that in addition to the conclusions of the lemma, $p(x) > p_1(x)$ almost everywhere.

In fact, we will use the following stronger form:

Lemma 3. *If T_σ is an ergodic $G-$extension of (T, X, μ), then given A and $B \subset X$ of equal positive measure, $g : A \to G$ measurable, $p_1 : A \to \mathbb{N}$, measurable, and a neighborhood U of e_G, there is a measurable $p : A \to \mathbb{N}$, with $p(x) > p_1(x)$ almost everywhere, such that $T^p : A \to B$ is an isomorphism, and $\sigma(x, p(x)) g(x)^{-1} \in U$ almost everywhere.*

Proof. Choose a neighborhood V of e_G such that $VV^{-1} \subset U$. Partition A into measurable sets $\{A_i\}_{i=1}^{\infty}$ such that for each i, there is some $g_i \in G$ such that $g(A_i) \subset Vg_i$. Applying Lemma 2, for each i, choose a measurable function $q_i : A_i \to \mathbb{N}$ with $q_i > p_1$ on A_i so that $T^{q_i} : A_i \to B$ is an isomorphism and $\sigma(x, q_i(x)) \in Vg_i$, almost everywhere in A_i, and so that the sets $\{T^{q_i} A_i\}_i$ are pairwise disjoint. Then for each $x \in A_i$, we have $\sigma(x, p(x)) \in Vg_i$. But $g(x) \in Vg_i$, so $\sigma(x, q_i(x)) g(x)^{-1} \in (Vg_i)(Vg_i)^{-1} = VV^{-1} \subset U$. Letting $p = \bigcup_{i=1}^{\infty} p_i$ completes the proof. □

Lemma 4. *Let G be a locally compact second countable group, and let T_σ be a $G-$extension of the aperiodic automorphism $(T, X, \mathcal{B}, \mu)$. Let $\{U_k\}_{k=1}^{\infty}$ be a neighborhood base for G at e_G. Then there is a sequence $\{\mathcal{C}_k\}_{k=1}^{\infty}$ of castles, where the towers of $\mathcal{C}_k$ all have height h_k, such that:*

(1) *for each k, $\mathcal{C}_k \leq \mathcal{C}_{k+1}$;*
(2) $\mu\left(\bigcup_{k=1}^{\infty} |\mathcal{C}_k|\right) = 1$;
(3) $\bigcup_{k=1}^{\infty} L(\mathcal{C}_k)$ *generates $\mathcal{B}$;*
(4) *for each tower $\mathcal{T}$ in $\mathcal{C}_k$ with base A, and each pair of levels $T^i A$ and $T^j A$ in $\mathcal{T}$, where $1 \leq i < j \leq h_k$, there is some $g \in G$ so that for all $x \in A$, $\sigma(x, j - i) \in U_k g$.*

Proof. Fix a sequence of finite partitions $\mathcal{P}_k \uparrow \mathcal{B}$ on X and a sequence $\varepsilon_k \downarrow 0$ with $\sum_k \varepsilon_k < 1$. Choose a sequence of towers $\mathcal{T}_k$ for T with residual sets of measure less than ε_k such that for each k, $|\mathcal{T}_k| \subset |\mathcal{T}_{k+1}|^o$. Denote the base of $\mathcal{T}_k$ by B_k and its height by h_k. Choose compact $K_1 \subset G$ so that if

$$B_1' = \left\{x \in B_1 \mid (\forall i, j \in \{0, ..., h_1 - 1\}) \, \sigma\left(T^i x, j - i\right) \in K_1\right\}$$

then $\mu(B_1') > (1 - \varepsilon_1) \mu(B_1)$. Let $\mathcal{T}_1'$ be the portion of $\mathcal{T}_1$ over B_1'. That is, $\mathcal{T}_1' = \left\{T_1^i B_1'\right\}_{i=0}^{h_1-1}$. Partition K_1 into sets $\{K_{1,i}\}_{i=1}^{s_1}$ so that for each $i = 1, ..., s_1$ there exists $g_{1,i} \in G$ with $K_{1,i} \subset U_1 g_{1,i}$. Let $\mathcal{K}_1 : G \to G$ be given by

$$\mathcal{K}_1(g) = \left\{ \begin{array}{l} g_{1,i}, \text{ if } g \in K_{1,i} \\ e_G \text{ if } g \in G \backslash K_1 \end{array} \right\}.$$

Let $\mathcal{Q}_1$ be the partition of B_1' according to the values of $\left\{\mathcal{K}_1\left(\sigma\left(T^i x, j - i\right)\right)\right\}_{i,j=0}^{h_1-1}$ and the values $\left\{\mathcal{P}_1\left(T^i(x)\right)\right\}_{i=0}^{h_1-1}$. ($\mathcal{P}_1(y)$ denotes the partition element containing y). We denote the resulting castle $(\mathcal{T}_1')_{\mathcal{Q}_1}$ by $\mathcal{C}_1'$.

Next consider $\mathcal{T}_2$ with base B_2 and height h_2. Choose compact $K_2 \subset G$ so that if

$$B_2' = \left\{x \in B_2 \mid (\forall i, j \in \{0, ..., h_2 - 1\}) \, \sigma\left(T^i x, j - i\right) \in K_2\right\}$$

then $\mu(B_2') > (1 - \varepsilon_2) \mu(B_2)$. Let $\mathcal{T}_2'$ be the the portion of $\mathcal{T}_2$ over B_2'. Partition K_2 into sets $\{K_{2,i}\}_{i=0}^{s_2}$ so that for each $i = 1, ..., s_2$ there exists $g_{2,i} \in G$ with

$K_{2,i} \subset U_2 g_{2,i}$. Define $\mathcal{K}_2 : G \to G$ analogously to $\mathcal{K}_1$. Let $\mathcal{P}_2' = \mathcal{P}_2 \vee \mathcal{P}\left(\mathcal{C}_1'\right)$, and let $\mathcal{Q}_2$ be the partition of B_2' according to the values of

$$\left\{\mathcal{K}_2\left(\sigma\left(T^i x, j-i\right)\right)\right\}_{i,j=0}^{h_2-1} \vee \left\{\mathcal{P}_2'\left(T^i\left(x\right)\right)\right\}_{i=0}^{h_2-1}.$$

This gives a castle $\mathcal{C}_2' = \left(\mathcal{T}_2'\right)_{\mathcal{Q}_2}$.

Repeating this process produces a sequence of castles $\mathcal{C}_k'$. To obtain condition 1, we restrict each $\mathcal{C}_k'$ to the set $\bigcap_{j=k+1}^{\infty}\left|\mathcal{C}_j'\right|$. That is, for each k and each level A' of $\mathcal{C}_k'$, we replace A' by the set $A = A' \cap \left(\bigcap_{j=k+1}^{\infty}\left|\mathcal{C}_j'\right|\right)$. The resulting set of levels is a castle $\mathcal{C}_k$, and these castles satisfy the conclusions of the lemma. $\square$

3. The main result

We now give the proof of Theorem 1.

Proof. Suppose that T_σ is an ergodic $G-$extension of (T, X, μ) and $T'_{\sigma'}$ is a $G-$extension of the aperiodic (T', X', μ'). Fix a neighborhood U of e_G, which we may assume to be compact. We will obtain the desired relative speedup of T_σ and the $G-$isomorphism from it to T'_σ as limits of a sequence of partially defined speedups and isomorphisms.

Let δ be a complete, right-invariant metric on G compatible with the topology of G. (We note that, while such a metric must exist, there need not be a complete, two-sided invariant metric compatible with the topology. See [B].) Fix $\varepsilon > 0$ so that $\bar{B}\left(\varepsilon, e_G\right)$, the closed $\delta-$ball of radius ε centered at e_G, is compact and contained in U. Fix a sequence $\varepsilon_k \downarrow 0$ with $\sum_{k=1}^{\infty}\varepsilon_k < \frac{\varepsilon}{3}$. For each k choose a compact neighborhood U_k of e_G so that $U_k U_k^{-1} \subset B\left(\varepsilon_k, e_G\right)$. Choose a sequence of castles $\{\mathcal{C}_k'\}_{k=1}^{\infty}$ for $T'_{\sigma'}$ as in Lemma 4 with respect to these U_k. Denote the towers and levels of these castles by $\mathcal{C}_k' = \left\{\mathcal{T}_{k,j}'\right\}_j$ and $\mathcal{T}_{k,j}' = \left\{A_{k,j,i}'\right\}_i$. In particular, Lemma 4 gives us, for all $i \in \{1, 2, ..., h_k - 1\}$ and for all levels $A_{k,j,i}'$ in $\mathcal{C}_k'$, an element $g_{k,j,i} \in G$ so that for all $x' \in A_{k,j,1}'$, $\sigma'\left(x', i\right) \in U_k g_{k,j,i}$.

Make a copy $\mathcal{C}_1$ of $\mathcal{C}_1'$ in X. That is, choose pairwise-disjoint sets $A_{1,j,i} \in \mathcal{B}$ corresponding to the levels of $\mathcal{C}_1'$ such that, for each j and i, $\mu\left(A_{1,j,i}\right) = \mu'\left(A_{1,j,i}'\right)$. Fix j and an isomorphism $\phi_{1,j} : A_{1,j,1} \to A_{1,j,1}'$.

Applying Lemma 3 repeatedly, we obtain functions $q_i : A_{1,j,1} \to \mathbb{N}$ with $q_i > q_{i-1}$ so that $T^{q_i} : A_1 \to A_i$ isomorphically, and for almost every $x \in A_1$ and every i,

$$\sigma\left(x, q_i\left(x\right)\right)\left(\sigma'\left(\phi_{1,j}\left(x\right), i\right)\right)^{-1} \in B\left(\varepsilon_1, e_G\right). \tag{3.1}$$

For each $i \in [1, h_1 - 1]$, let $p_i : A_i \to \mathbb{N}$ be given by setting

$$p_i\left(x\right) = q_{i+1}\left(T^{-q_i}\left(x\right)\right) - q_i\left(T^{-q_i}\left(x\right)\right).$$

Then, letting $p = \bigcup_{i=1}^{h_1-1} p_i$, we obtain a partially defined speedup $T_1 := T^p$ of T, defined on $\left|\mathcal{T}_{1,j}\right|^o$, for which $\mathcal{T}_{1,j}$ is a tower. This construction also yields a partially defined cocyle σ_1 for T_1, which is defined at (x, n) whenever $x \in \left|\mathcal{T}_{1,j}\right| \cap T_1^{-n}\left|\mathcal{T}_{1,j}\right|$.

Extend $\phi_{1,j}$ to $\left|\mathcal{T}_{1,j}\right|$ so that on $\left|\mathcal{T}_{1,j}\right|^o$

$$\phi_{1,j} T_1\left(x\right) = T'\phi_{1,j}\left(x\right) \text{ a.e.}$$

In particular, for each i, $\phi_{1,j}(A_{1,j,i}) = A'_{1,j,i}$. Define $\alpha_{1,j} : |\mathcal{T}_{1,j}| \to G$ by setting, for each $x \in A_{1,j,i}$,

$$\alpha_{1,j}(x) = \sigma'\left(\phi_{1,j}(x), -i\right)^{-1} \sigma_1(x, -i).$$

Repeating this construction on each tower of $\mathcal{C}_1$, we set $\phi_1 = \bigcup_j \phi_{1,j}$ to obtain an isomorphism from $|\mathcal{C}_1|$ to $|\mathcal{C}'_1|$ intertwining T_1 and T'. Similarly, we let $\alpha_1 = \bigcup_j \alpha_{1,j}$ and extend α_1 to X by setting $\alpha_1(x) = e_G$ for $x \in X \setminus |\mathcal{C}_1|$. We then see that the map $(x, g) \mapsto (\phi_1(x), \alpha_1(x) g)$ is a $G-$isomorphism from $(T_1)_{\sigma_1}$ to $T'_{\sigma'}$, insofar as these maps are defined, which is to say on $|\mathcal{C}_1|^o$. In other words, for all (x, n) in the domain of σ_1,

$$\sigma_1^{\alpha_1}(x, n) := \alpha_1\left((T_1)^n x\right) \sigma_1(x, n) \alpha_1(x)^{-1} = \sigma'(\phi_1(x), n). \tag{3.2}$$

We also see that, because of condition (3.1) and the right-invariance of δ, we have for all $x \in X$,

$$\delta(\alpha_1(x), \varepsilon_G) < \varepsilon_1. \tag{3.3}$$

(We note that in the above construction the approximate constancy of σ' on the levels of $\mathcal{C}'_1$ was not used.)

Now we show how to iterate this construction to complete the proof of the theorem. Fix an increasing sequence of finite partitions $\{\mathcal{P}_k\}_{k=1}^{\infty}$ of X that generate $\mathcal{B}$. Choose n_2 so that the partition $\phi_1(\mathcal{P}_1)$ is approximated to within $\frac{1}{2}$ (in the partition metric) by the levels of $\mathcal{C}'_{n_2}$. The index n_2 must also be chosen so that ε_{n_2} is small enough to meet an additional condition, which we will describe at the end of the proof. For notational convenience re-index $\mathcal{C}'_{n_2}$ and refer to it as $\mathcal{C}'_2$, and do the same with ε_{n_2}, U_{n_2}, and so on. Let $\mathcal{C}_2$ denote a copy of $\mathcal{C}'_2$ which is the image of $\mathcal{C}'_2$ under ϕ_1^{-1}. That is, for each level $A'_{2,j,i}$ of $\mathcal{C}'_2$ contained in $|\mathcal{C}'_1|$, the corresponding level $A_{2,j,i}$ of $\mathcal{C}_2$ is given by $A_{2,j,i} = \phi_1^{-1}\left(A'_{2,j,i}\right)$. Additional subsets of X are chosen to serve as $A_{2,j,i}$ when $A'_{2,j,i}$ is not contained in $|\mathcal{C}'_1|$.

Our goal is to extend T_1 to a transformation T_2 on $|\mathcal{C}_2|^o$, so that T_2 is again a partially defined speedup of T, with an associated cocycle σ_2. We will also modify α_1 to a function $\alpha_2 : X \to G$ so that on $|\mathcal{C}_2|$, α_2 serves as a transfer function for a $G-$isomorphism between $(T_2)_{\sigma_2}$ and $T'_{\sigma'}$.

Note that since U_2 is compact, and there are only finitely many towers in $\mathcal{C}'_2$, there is a compact set K so that for all (x', n) with $x' \in |\mathcal{C}'_2| \cap T'^{-n}|\mathcal{C}'_2|$, $\sigma'(x', n) \in K$. Choose $\xi_2 \in (0, \varepsilon_2)$ so that if $a, b \in K$, and $\delta(a, a') < \xi_2$, then $\delta(ba, ba') < \varepsilon_2$. (This is possible by invoking the uniform continuity of the group multiplication on WK, where W is a compact neighborhood of e_G.)

Fix a tower $\mathcal{T}_{2,j}$ in $\mathcal{C}_2$ and suppose that $|\mathcal{T}_{2,j}| \cap |\mathcal{C}_1| \neq \emptyset$. Let $\phi_2 : A_{2,j,1} \to A'_{2,j,1}$ be an isomorphism. For each level $A_{2,j,m} \subset |\mathcal{T}_{2,j}|$ define a function $q_m : A_{2,j,1} \to \mathbb{N}$ so that $T^{q_m} : A_{2,j,1} \to A_{2,j,m}$ isomorphically, and for almost all $x \in A_{2,j,1}$,

$$\sigma'\left(\phi_{2,j}(x), m\right) \sigma(x, q_m(x))^{-1} \in B(\xi_2, e_G).$$

The q_m are chosen so that $q_{m+1} > q_m$ and so that, as before, defining p_2 on each $A_{2,j,m}$ by

$$p_2(x) = q_{m+1}\left(T^{-q_m}(x)\right) - q_m\left(T^{-q_m}(x)\right)$$

the transformation $T_2(x) = T^{p_2}(x)$ is a speedup of T and agrees with T_1 on its domain. This can be done by repeated application of Lemma 3. Explicitly, if

$A_{2,j,1} \not\subset |\mathcal{C}_1|$, then by Lemma 3 there is a function $q_1 : A_{2,j,1} \to \mathbb{N}$ so that $T^{q_1} : A_{2,j,1} \to A_{2,j,2}$ isomorphically, and for almost all $x \in A_{2,j,1}$

$$\sigma' \left(\phi_{2,j}(x), 1\right) \sigma(x, q_1(x))^{-1} \in B(\xi_2, e_G).$$

If $A_{2,j,2} \not\subset |\mathcal{C}_1|$, then we choose $q_2 > q_1$ on $A_{2,j,1}$ so that $T^{q_2} : A_{2,j,1} \to A_{2,j,3}$ isomorphically, and for almost all $x \in A_{2,j,1}$

$$\sigma' \left(\phi_{2,j}(x), 2\right) \sigma(x, q_2(x))^{-1} \in B(\xi_2, e_G).$$

We continue in this way until (unless) we first arrive at a level $A_{2,j,m} \subset |\mathcal{C}_1|$. There T_1 is already defined, and we let $q_{m+1} = q_m + p_1$. We continue this way until we reach the top level $A_{2,j,m+h_1-1}$ of this $\mathcal{C}_1$−column. If there is another level $A_{2,j,m+h_1}$ of $|\mathcal{T}_{2,j}|$, we define q_{m+h_1} as before and continue until all levels of $|\mathcal{T}_{2,j}|$ have been addressed.

Having defined T_2 on $|\mathcal{T}_{2,j}|^o$, $\phi_{2,j}$ is then extendible uniquely to $|\mathcal{T}_{2,j}|$ by the requirement that for almost all $x \in |\mathcal{T}_{2,j}|^o$,

$$\phi_{2,j}(T_2 x) = T' \left(\phi_{2,j} x\right).$$

Let σ_2 denote the cocyle determined by T_2 and σ, which is defined for pairs (x, n), where $x \in |\mathcal{T}_{2,j}| \cap T_2^{-n} |\mathcal{T}_{2,j}|$. We define $\alpha_2 : |\mathcal{T}_{2,j}| \to G$ in two stages. First, for $x \in A_{2,j,m} \subset |\mathcal{T}_{2,j}| \cap |\mathcal{C}_1|$, if $x \in A_{1,j,l+1}$ (that is, x is in the $(l+1)^{st}$ level of $\mathcal{C}_1$), then we let

$$\bar{\alpha}_1(x) = \sigma' \left(\phi_{2,j}(x), -l\right)^{-1} \sigma_2^{a_1}(x, -l).$$

We set $\bar{\alpha}_1(x) = e_G$ on $|\mathcal{T}_{2,j}| \setminus |\mathcal{C}_1|$. We see that for $x \in |\mathcal{T}_{2,j}| \cap |\mathcal{C}_1|$,

$$\begin{aligned} \delta(\bar{\alpha}_1(x), e_G) &= \delta\left(\sigma' \left(\phi_{2,j}(x), -l\right)^{-1}, \sigma_2^{a_1}(x, -l)^{-1}\right) \\ &\le \delta\left(\sigma' \left(\phi_{2,j}(x), -l\right)^{-1}, \sigma'(\phi_1(x), -l)\right) \\ &\quad +\delta\left(\sigma'(\phi_1(x), -l), \sigma_2^{a_1}(x, -l)^{-1}\right) \\ &= \delta\left(\sigma' \left(\phi_{2,j}(x), -l\right)^{-1}, \sigma'(\phi_1(x), -l)\right) \\ &\le 2\varepsilon_2 \end{aligned}$$

where the last inequality follows from the condition on the near constancy of σ, on the columns of $\mathcal{C}_2'$. Thus $\delta(\bar{\alpha}_1(x)\alpha(x), \alpha(x)) = \delta(\bar{\alpha}_1(x), e_G) \le 2\varepsilon_2$. Moreover, on a $\mathcal{C}_1$−column contained in $\mathcal{T}_{2,j}$, the map $(x, g) \mapsto \left(\phi_{2,j}(x), \bar{\alpha}_1(x)\alpha_1(x) g\right)$ is a G−isomorphism from $(T_2)_{\sigma_2}$ to $T'_{\sigma'}$.

Now for any $x \in A_{2,j,m+1} \subset |\mathcal{T}_{2,j}|$ let

$$\tilde{\alpha}_1(x) = \sigma' \left(\phi_{2,j}(x), -m\right)^{-1} \sigma_2^{\bar{\alpha}_1 \alpha_1}(x, -m).$$

We see that $\delta(\tilde{\alpha}_1(x), e_G) < \varepsilon_2$. Indeed, if $x \in |\mathcal{T}_{2,j}| \setminus |\mathcal{C}_1|$, or if x is in $|\mathcal{T}_{2,j}|$ and in the base of $\mathcal{C}_1$, then this is immediate from the construction. On the other hand, suppose $x \in |\mathcal{C}_1|$ but not in the base of $\mathcal{C}_1$. Say x is in $A_{2,j,m+1}$ and in the $(l+1)^{st}$ level of $\mathcal{C}_1$. Then we have

$$\delta(\tilde{\alpha}_1(x), e_G) = \delta\left(\sigma' \left(\phi_{2,j}(x), -m\right)^{-1}, \sigma_2^{\bar{\alpha}_1 \alpha_1}(x, -m)^{-1}\right).$$

But

$$\sigma' \left(\phi_{2,j}(x), -m\right)^{-1} = \sigma' \left(\phi_{2,j}(x), -l\right)^{-1} \sigma' \left(T'^{-l}\left(\phi_{2,j}(x)\right), -m\right)^{-1}$$

and
$$\sigma_2^{\bar{\alpha}_1\alpha_1}(x,-m)^{-1} = \sigma_2^{\bar{\alpha}_1\alpha_1}(x,-l)^{-1}\,\sigma_2^{\bar{\alpha}_1\alpha_1}\left(T_2^{-l}(x),-m\right)^{-1}.$$
Furthermore,
$$\sigma'\left(\phi_{2,j}(x),-l\right) = \sigma_2^{\bar{\alpha}_1\alpha_1}(x,-l)$$
and
$$\delta\left(\sigma'\left(T'^{-l}\left(\phi_{2,j}(x)\right),-m\right)^{-1},\sigma_2^{\bar{\alpha}_1\alpha_1}\left(T_2^{-l}(x),-m\right)^{-1}\right) < \xi_2$$
so by the choice of ξ_2 we conclude that
$$\delta\left(\tilde{\alpha}_1(x),e_G\right) \leq \varepsilon_2.$$

Now set $\alpha_2(x) = \tilde{\alpha}_1(x)\,\bar{\alpha}_1(x)\,\alpha_1(x)$ and observe that
$$\delta\left(\alpha_2(x),\alpha_1(x)\right) \leq 3\varepsilon_2$$
(using the right invariance of δ). The map $(x,g) \mapsto \left(\phi_{2,j}(x),\alpha_2(x)\,g\right)$ is a $G-$isomorphism from $(T_2)_{\sigma_2}$ to $T'_{\sigma'}$ on all of $|\mathcal{T}_{2,j}|$.

Perform this construction on each tower of $\mathcal{C}_2$ which meets $|\mathcal{C}_1|$. If $\mathcal{T}_{2,j}$ is a tower of $\mathcal{C}_2$ that does not meet $|\mathcal{C}_1|$, then employ the simpler construction that was used in the first stage of the proof to define T_2 and α_2 on such a tower. Setting $\alpha_2(x) = e_G$ on $X\setminus|\mathcal{C}_2|$ completes the second stage of the proof.

This procedure can be repeated indefinitely to produce a sequence of castles $\mathcal{C}_k$ in X for partially defined transformations T_k, where the levels of $\mathcal{C}_k$ approximate the partition $\mathcal{P}_{k-1}$ to within $\frac{1}{k}$, so that each T_k is a speedup of T defined on $|\mathcal{C}_k|^o$, each T_{k+1} extends T_k, and the transformation $\bar{T} = \bigcup_k T_k$ is a speedup of T defined almost everywhere. Let $\bar{\sigma}$ denote the cocycle for $\bar{T}$ that arises from σ.

The construction also produces a sequence of isomorphisms $\phi_k : |\mathcal{C}_k| \to |\mathcal{C}'_k|$ that intertwine T_k and T'. In addition, it produces a sequence of functions $\alpha_k : X \to G$ so that for each k, $\delta(\alpha_{k+1},\alpha_k) \leq 3\varepsilon_{k+1}$ and so that the map $(x,g) \mapsto (\phi_k x, \alpha_k g)$ is a $G-$isomorphism between $(T_k)_{\sigma_k}$ and $T'_{\sigma'}$ on $|\mathcal{C}_k|$. Since δ is complete we see that the sequence α_k converges uniformly to a function α, such that for almost all x, $\delta(\alpha(x),e_g) \leq \varepsilon$, and hence $\alpha(x) \in U$.

In the contruction of the ϕ_k we observe that each ϕ_{k+1} agrees set-wise with ϕ_k on the levels of $\mathcal{C}_k$. Since the $\sigma-$algebras $\mathcal{B}_k$ generated by the levels of $\mathcal{C}_k$ increase to the full $\sigma-$algebra, the maps ϕ_k determine an isomorphism ϕ between $\bar{T}$ and T' which, for each k, agrees set-wise with ϕ_k on $\mathcal{B}_k$.

Now we confirm that the map $(x,g) \mapsto (\phi x, \alpha(x)\,g)$ is a $G-$isomorphism from $\bar{T}_{\bar{\sigma}}$ to $T'_{\sigma'}$. We want to establish that for each n,
$$\begin{aligned} \sigma'(\phi x,n) &= \alpha\left(\bar{T}^n x\right)\bar{\sigma}(x,n)\,\alpha(x)^{-1} \text{ a.e.} \\ &= \bar{\sigma}^{\alpha}(x,n). \end{aligned}$$
Fix $n \in \mathbb{Z}$ and $\eta > 0$. For almost every x, if k is sufficiently large, then
$$\bar{T}^n x = T_k^n x,$$
$$\bar{\sigma}(x,n) = \sigma_k(x,n)$$
and
$$\delta\left(\alpha(x),\alpha_k(x)\right) \leq \eta.$$
Furthermore, since the points $\phi(x)$ and $\phi_k(x)$ are in the same level of $\mathcal{C}'_k$, the approximate constancy of $\sigma'(\cdot,n)$ on such a level gives
$$\delta\left(\sigma'(\phi x,n),\sigma'(\phi_k x,n)\right) \leq \eta.$$

But we know that

$$\begin{aligned}\sigma'\left(\phi_k x, n\right) &= \alpha_k\left(T_k^n x\right)\sigma_k\left(x,n\right)\alpha_k\left(x\right)^{-1} \text{ a.e.}\\ &= \alpha_k\left(\bar{T}^n x\right)\bar{\sigma}\left(x,n\right)\alpha_k\left(x\right)^{-1}.\end{aligned}$$

We only need to conclude that this value is close to $\bar{\sigma}^{\alpha}\left(x,n\right)$.

Now we describe the additional condition according to which the subsequence $\varepsilon_{n_2}, \varepsilon_{n_3}, ...$ (relabeled $\varepsilon_2, \varepsilon_3, ...$) must be chosen. For each $k \geq 1$, there is a fixed compact set K containing all the values of α_k and all the values of $\sigma_k\left(x,t\right)$, for $x \in \left|\mathcal{C}_k\right| \cap T_k^{-t}\left|\mathcal{C}_k\right|$. (Recall that all the values of α_k lie in $\bar{B}\left(\varepsilon, e_G\right)$, which is compact.) Regardless of how the ε_k will be chosen, we will have, for all k and x,

$$\delta\left(\alpha_k\left(x\right), \alpha\left(x\right)\right) < \sum_{m=k+1}^{\infty}\varepsilon_m.$$

The additional condition we impose on the ε_k is that this sum is so small as to ensure that for all $x \in \left|\mathcal{C}_k\right| \cap T_k^{-t}\left|\mathcal{C}_k\right|$,

$$\delta\left(\alpha_k\left(T_k^t x\right)\sigma_k\left(x,t\right)\alpha_k\left(x\right)^{-1}, \alpha\left(\bar{T}^t x\right)\sigma_k\left(x,t\right)\alpha\left(x\right)^{-1}\right) < \frac{1}{k}.$$

If this is done, then for the given n and η, arguing with sufficently large k, we can conclude that

$$\delta\left(\sigma'\left(\phi x, n\right), \alpha\left(\bar{T}^n x\right)\bar{\sigma}\left(x,n\right)\alpha\left(x\right)^{-1}\right) < \eta + \frac{1}{k}.$$

Since η is arbitrary, we have the desired equality. □

We note that in the case that G is a discrete group, we immediately obtain the following stronger result:

Corollary 1. *Let T_σ and $T'_{\sigma'}$ be $G-$extensions of T and T', where G is a finite or countable group. Suppose that T_σ is ergodic and T' is aperiodic. Then there is a relative speedup of T_σ which is $G-$isomorphic to $T'_{\sigma'}$ by a relative isomorphism whose transfer function α satisfies $\alpha\left(x\right) = e_G$ almost everywhere.*

4. Finite and countable extensions

We now turn to the analysis of speedups of $n-$point extensions. First we introduce a simplification of some of our notation. Given an automorphism T and a cocycle σ taking values in a group G, we will denote the $G-$extension T_σ more simply by the single letter S, and in general, $G-$extensions $T'_{\sigma'}$ or $\left(T_1\right)_{\sigma_1}$ will be denoted S' and S_1, and so on.

Now fix an integer $n > 1$. Form the measure space $\left\{[n], \mathcal{P}\left([n]\right), p\right\}$ where $[n] = \{1, ..., n\}$, and $p\left(\{i\}\right) = \frac{1}{n}$, for each i. Making use of the natural action of the symmetric group $\mathcal{S}_n$ on $[n]$, each cocycle σ for T taking values in $\mathcal{S}_n$ determines an automorphism U of $\left\{X \times [n], \mathcal{B} \times \mathcal{C}, \mu \times p\right\}$ which has T as a factor. Namely, we have the automorphism U given by

$$U^n\left(x,i\right) = \left(T^n x, \sigma\left(x,n\right)\left(i\right)\right).$$

We refer to U as an $n-$point extension of T. (Since we will only consider ergodic $n-$point extensions, we may restrict ourselves to the uniform measure p). We

will use the same sort of notational convention as above: the $n-$point extensions associated with pairs (T',σ') and (T_1,σ_1) will be written U' and U_1, and so on.

Given a pair of $n-$point extensions U_1 and U_2 on spaces $X_1\times[n]$ and $X_2\times[n]$, we say U_1 is relatively isomorphic to U_2 if there is an isomorphism Φ from U_1 to U_2 that preserves the fibers of these extensions. That is, there is an isomorphism of the form

$$(x,i)\mapsto(\phi(x),\alpha(x)(i))$$

where $\alpha:X_1\to\mathcal{S}_n$. Equivalently, these extensions are relatively isomorphic if there is an isomorphism ϕ from T_1 to T_2 and a function $\alpha:X_1\to\mathcal{S}_n$ such that

$$\sigma_2(\phi x,n)=\alpha\left(T_1^n(x)\right)\sigma_1(x,n)\,\alpha(x)^{-1}$$

which is exactly the condition that the $\mathcal{S}_n-$extensions S_1 and S_2 are $\mathcal{S}_n-$isomorphic. We also note that every speedup of T_1 (or equivalently, every $\mathcal{S}_n-$speedup of S_1) corresponds to a speedup of U_1 relative to T_1.

Given $n-$point extensions U_1 and U_2, let us write $U_1\rightsquigarrow U_2$ when there is a speedup of U_1 relative to T_1 which is relatively isomorphic to U_2. This relation is evidently transitive and apparently asymmetric; there is no reason to suppose that $U_1\rightsquigarrow U_2$ implies $U_2\rightsquigarrow U_1$. In the case of ergodic finite group extensions, however, (as well as for more general locally compact second countable group extensions), we have just seen that it is symmetric, and in fact for each locally compact second countable group G there is only one equivalence class of ergodic $G-$extensions. But, for general ergodic $n-$point extensions, we will see that the relation is indeed asymmetric. This is due to the fact that the associated $\mathcal{S}_n-$ extensions, of which the ergodic $n-$point extensions are factors, need not themselves be ergodic. By examining the ergodic components of these $\mathcal{S}_n-$ extensions, we will obtain a characterization of this relation in other terms, and we will give an explicit example to illustrate its asymmetry.

Fix an automorphism T and an $\mathcal{S}_n-$cocycle σ. Let S be the associated $\mathcal{S}_n-$extension of T. We associate to the pair (T,σ) a conjugacy class of subgoups of $\mathcal{S}_n$ that will be the basis of the characterization. We recall the discussion that can be found in [G]: Let C be an ergodic component of S. For each $x\in X$, let $C_x=\{\gamma\in\mathcal{S}_n:(x,\gamma)\in C\}$. (By the ergodicity of T, $|C_x|\geq 1$ is a constant). Then if $\beta:X\to\mathcal{S}_n$ is any measurable function such that $(x,\beta(x))\in C$ almost everywhere, there is a subgroup G of $\mathcal{S}_n$ such that the sets $\beta(x)^{-1}C_x$ are almost all equal to G. Moreover, letting $\alpha(x)=\beta(x)^{-1}$, and defining a new cocycle by $\sigma'(x,n)=\alpha(T^nx)\,\sigma(x,n)\,\alpha(x)^{-1}$, we get a new $\mathcal{S}_n-$extension S' that is $\mathcal{S}_n-$isomorphic to S. The map

$$(x,\gamma)\mapsto(x,\alpha(x)\gamma)$$

is an $\mathcal{S}_n-$isomorphism from S to S', which carries C to $X\times G$, on which S' is ergodic. In summary: σ is cohomologous to a $G-$valued cocycle σ' yielding an $\mathcal{S}_n-$extension S' that has $X\times G$ as an ergodic component. (In particular, the values of σ' lie in G). The groups G that fit this description form a conjugacy class of subgroups of $\mathcal{S}_n$. We denote this conjugacy class by $gp(T,\sigma)$.

We can easily extend the above discussion to a slightly more general context: Suppose that, for some subgroup $G\subset\mathcal{S}_n$, $X\times G$ is $S-$invariant, but is not an ergodic component of S. Then we can apply the above arguments to an ergodic component of S that is contained in $X\times G$, and conclude that there is a subgroup

$H \subset G$ and a cocycle σ' cohomologous to σ via a $G-$valued transfer function, so that $X \times H$ is an ergodic component of the $\mathcal{S}_n-$extension S' associated with σ'.

For brevity, when $X \times G$ is an ergodic component of an $\mathcal{S}_n-$extension S, we will say that σ is $G-$ergodic for T. (This is equivalent to the "$G-$interchange property" that Gerber introduced in [G]).

It is clear that $gp\,(T, \sigma)$ is an invariant of factor isomorphism. In [G] Gerber showed that it is a complete invariant for factor orbit equivalence of ergodic $n-$point extensions. In connection with speedups, we now prove:

Theorem 2. *Let U_1 and U_2 be ergodic $n-$point extensions of transformations (T_1, X_1) and (T_2, X_2) by $\mathcal{S}_n-$valued cocycles σ_1 and σ_2. Then $U_1 \rightsquigarrow U_2$ if and only if for some $G_1 \in gp\,(T_1, \sigma_1)$ (and hence for every $G_1 \in gp\,(T_1, \sigma_1)$), there exists $G_2 \in gp\,(T_2, \sigma_2)$ such that $G_2 \subset G_1$.*

Proof. Suppose first that for some $G_1 \in gp\,(T_1, \sigma_1)$, there exists $G_2 \in gp\,(T_2, \sigma_2)$ such that $G_2 \subset G_1$. By the above discussion, for each $i = 1, 2$, there is a cocycle σ_i' cohomologous to σ_i, so that $X_i \times G_i$ is an ergodic component of $(T_i)_{\sigma_i}$. Therefore, without loss of generality, we may assume from the start that each (T_i, σ_i) has this property.

S_1 induces an ergodic transformation $(S_1)_{X_1 \times G_2}$ on $(X_1 \times G_2)$ and, for each $(x, g_2) \in X_1 \times G_2$ we let $j = j\,(x, g_2)$ denote the first return time of (x, g_2) to $X_1 \times G_1$ under S_1. But $j\,(x, g_2)$ depends only on x since, for all $(x, g_2) \in X_1 \times G_2$ and all $n \in \mathbb{N}$,

$$S_1^n\,(x, g_2) \in (X_1 \times G_2) \iff \sigma_1\,(x, n)\, g_2 \in G_2 \iff \sigma_1\,(x, n) \in G_2$$

so that these conditions do not depend on g_2. We can then write $j\,(x, g_2) = j\,(x)$, and the induced automorphism

$$(S_1)_{X_1 \times G_2} = S_1^j \restriction_{X_1 \times G_2}$$

is an ergodic G_2-extension of the speedup T_1^j of T_1.

Applying Theorem 1, we know that there is a G_2-speedup $\left(S_1^j \restriction_{X_1 \times G_2}\right)^k$ of $S_1^j \restriction_{X_1 \times G_2}$, that is G_2-isomorphic to $S_2 \restriction_{X_2 \times G_2}$. This gives us a speedup $\left(T_1^j\right)^k = T_1^l$ of T_1, and $\left(S_1^j \restriction_{X_1 \times G_2}\right)^k = S_1^l \restriction_{X_1 \times G_2}$ is a G_2-extension of T_1^l.

There is an isomorphism $\phi : X_1 \to X_2$ and $\alpha : X_1 \to G_2$ so that

$$T_1^l \overset{\phi}{\approx} T_2 \text{ and } \sigma_2 \phi = \left(\sigma_1^{(l)}\right)^\alpha. \tag{4.1}$$

This construction gives us a speedup $U_1^l$of U_1 relative to T_1 and conditions (4.1) say that U_1^l is relatively isomorphic to U_2.

Now suppose that $U_1 \rightsquigarrow U_2$. Fix $G_1 \in gp\,(S_1)$. We want to show that there is a subgroup $G_2 \in sp\,(S_2)$ contained in G_1. As before, we may assume that $X_1 \times G_1$ is an ergodic component of S_1. The condition $U_1 \rightsquigarrow U_2$ tells us that there is an $\mathcal{S}_n-$speedup S_1^k of S_1 that is $\mathcal{S}_n-$isomorphic to S_2. Since $X_1 \times G_1$ is invariant for S_1, it remains so for S_1^k. Arguing as before, there is a subgroup $G_2 \subset G_1$ and a cocycle $\bar{\sigma}$ for T_1^k, cohomologous to $\sigma_1^{(k)}$, such that $X_1 \times G_2$ is an ergodic component of $\bar{S}$, where $\bar{S}$ is the $\mathcal{S}_n-$extension of T_1^k given by $\bar{\sigma}$. But, since $\bar{S}$ is relatively isomorphic to S_2, we must have $G_2 \in gp\,(T_2, \sigma_2)$. □

As an example, we consider a pair of 3−point extensions considered by Gerber in [G]. Let $\{\gamma_i\}_{i=1}^{6}$ be an enumeration of the symmetric group $\mathcal{S}_3$ so that $\{\gamma_i\}_{i=1}^{3}$ is the alternating group $\mathcal{A}_3$. Let (T_1, X_1) be the full 3−shift, with independent generator $\mathcal{P} = \{P_i\}_{i=1}^{3}$ and (T_2, X_2) the full 6−shift with independent generator $\mathcal{Q} = \{Q_i\}_{i=1}^{6}$. Define a cocycle σ_1 for T_1 by setting $\sigma(x, 1) = \gamma_i$ when $x \in P_i$. Define σ_2 for T_2 by setting $\sigma_2(x, 1) = \gamma_i$ when $x \in Q_i$. In [G] Gerber showed that $gp(S_1) = \{\mathcal{A}_3\}$ and $gp\{S_2\} = \{\mathcal{S}_3\}$ and that consequently S_1 and S_2 are not factor orbit equivalent. Using Theorem 2 we can conclude further that there is a factor speedup of S_2 that is factor isomorphic to S_1, but there is no factor speedup of S_1 that is factor isomorphic to S_2.

As a final application we observe how the classification of extensions changes when we pass to extensions that are as close as possible to the finite case. That is, we consider extensions with countable fibers, and allow only the smallest natural group of permutations on the fibers. Let p denote counting measure on $\mathbb{N}$ and $\mathcal{S}_f$ the group of finitely supported permutations of $\mathbb{N}$. Suppose that T is an automorphism of the Lebesgue probability space (X, B, μ), and σ is an $\mathcal{S}_f$−cocycle for T. Then as in the finite case, we obtain a countable extension U of T on $X \times \mathbb{N}$ given by

$$U^n(x, k) = (T^n x, \sigma(x, n)(k)).$$

Here we say that two such countable extensions U_1 and U_2 are relatively isomorphic if there is an isomorphism ϕ from T_1 to T_2 and a function $\alpha : X \to \mathcal{S}_f$ such that

$$\sigma_2(\phi x, n) = \alpha(T_1^n(x))\, \sigma_1(x, n)\, \alpha(x)^{-1}.$$

Since $\mathcal{S}_f$ is countable, Theorem 1 applies, and the analysis of finite extensions which was given above can easily be adapted to this case. Thus, to each ergodic countable extension U of T by an $\mathcal{S}_f$−valued cocycle σ, we associate a conjugacy class $gp(T, \sigma)$ of subgroups of $\mathcal{S}_f$, and the corresponding statement of Theorem 2 holds.

Proposition 1. *Given an ergodic T, there is an uncountable family of ergodic countable extensions of T, each of which acts on the fibers of the extension by finitely supported permuations of $\mathbb{N}$, and no one of which admits a relative speedup that is relatively isomorphic to another.*

Proof. We first construct an uncountable family of subgroups of $\mathcal{S}_f$ such that (i), each acts transitively on $\mathbb{N}$, and (ii), no conjugate of one contains another. Suppose $\mathcal{P} = \{A_k\}_{k=1}^{\infty}$ is a partition of $\mathbb{N}$ into two-element sets. Let

$$G_{\mathcal{P}} = \{\pi \in \mathcal{S}_f : (\forall k)(\exists j)\, \pi(A_k) = A_j\}.$$

It is clear that $G_{\mathcal{P}}$ acts transitively on $\mathbb{N}$. For $\xi \in \mathcal{S}_f$ we write $\xi\mathcal{P} = \{\xi A \mid A \in \mathcal{P}\}$ and observe that $\xi G_{\mathcal{P}} \xi^{-1} = G_{\xi\mathcal{P}}$. Hence, if $\mathcal{P}'$ is another partition of $\mathbb{N}$ into two-element sets and the symmetric difference of $\mathcal{P}$ and $\mathcal{P}'$ is infinite, then no conjugate of $G_{\mathcal{P}}$ can contain $G_{\mathcal{P}'}$. It is easy to construct an uncountable family $\{\mathcal{P}_i\}_{i\in I}$ of such partitions so that each pair has an infinite symmetric difference. The corresponding family of subgroups $\{G_{\mathcal{P}_i}\}_{i\in I}$ satisfies conditions (i) and (ii) above.

Each of the groups $G_{\mathcal{P}_i}$ is countable and amenable, and hence, by the result of Herman [H], for each G_{P_i} there is a cocycle σ_i for T for which the corresponding $G_{\mathcal{P}_i}$−extension S_i is ergodic. But since $G_{\mathcal{P}_i}$ acts transitively on $\mathbb{N}$, the corresponding countable extension U_i is also ergodic. Indeed, for all sets A and B contained in

X of positive measure, and all $l, m \in \mathbb{N}$, if we choose $\pi \in G_{P_i}$ such that $\pi(l) = m$, then Lemma 1 gives an $n' \in \mathbb{N}$ and $A' \subset A$ with $\mu(A') > 0$ such that $T^{n'}(A') \subset B$ and for all $x \in A'$, $\sigma(x, n') = \pi$. Condition (ii) on the groups $G_{\mathcal{P}_i}$ tells us that for all $i \neq j$ in I, no speedup of U_i relative to T is relatively isomorphic to U_j. $\square$

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